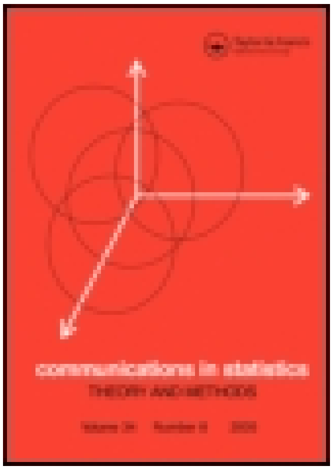

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A rank test for stochastically ordered alternatives

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A RANK TEST FOR STOCHASTICALLY ORDERED ALTERNATIVES

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ABSTRACT

A rank test based on the number of 'near–matches' among within–block rankings is proposed for stochastically ordered alternatives in a randomized block design with t treatments and b blocks. The asymptotic relative efficiency of this test with respect to the Page test is computed as number of blocks increases to infinity. A sequential analog of the above test procedure is also considered. A repeated significance test procedure is developed and average sample number is computed asymptotically under the null hypothesis as well as under a sequence of contiguous alternatives.

1. INTRODUCTION

The aim of this paper is to develop a new rank test to test the equality of t distribution functions against possible stochastic ordering among them. Consider

$$H_0 : F_1 = \ldots = F_t$$

against the ordered alternative

$$K: F_1 \leq F_2 \leq \ldots\ldots \leq F_t,$$

where at least one of the inequalities is strict. Let $X = (X_{ij})$, $i=1,\ldots,b$, $j = 1,\ldots,t$ denote the data. We assume that

$$X_{ij} = Z_{ij} + B_i, \quad i = 1,\ldots b, \quad j = 1,\ldots,t,$$

where $\{Z_{ij}\}$ and $\{B_i\}$, $i = 1,\ldots,b$, $j = 1,\ldots,t$, are mutually independent random variables (r.v.'s) and $\{Z_{1j},\ldots,Z_{bj}\}$ are r.v.'s with (d.f.) F_j and B_i's are identically distributed. This is a generalization of the fixed effects Analysis of Variance (ANOVA) model where $X_{ij} = Y_{ij} + d_j + b_i$, and $\{Y_{ij}\}$, $i = 1,\ldots,b$, $j = 1,\ldots,t$, are independent and identically distributed (i.i.d) r.v.'s with d.f. F and $\{d_j\}$ and $\{b_i\}$ are the nonrandom treatment and block effects respectively.

Several test procedures are available in the literature for the proposed hypothesis. Page's test [6] based on Spearman's rank correlation is a rank test based on within–block ranks whereas tests proposed by Hollander [3], Puri and Sen [8], Sen [11] are among–block rank tests. Each test has its merits and demerits and they have been discussed by Pirie [7], and Puri and Sen [9]. Indeed, Page's test is the locally most powerful rank test for Logistic distribution alternatives with linear trend. It should be emphasized that this optimality property is maintained only if the location trends d_j's are linear in j. A new rank test based on the idea of 'Near–Match' [4] is developed and the case of number of blocks going to infinity is discussed. The case of number of teatments going to infinity is discussed by Jammalamadaka et al. [5].

Let $(R_{i1},\ldots,R_{it})$ denote the within–block ranking of the i^{th} block. For any fixed non–negative integer $k(< \frac{t}{2})$ we say that a 'near–match' has occurred at the j^{th} place if $|R_{ij}-j| \leq k$. Let $M^b = M(k,t,b) = \sum_{i=1}^{b} M_i(k,t)$ where $M_i = M_i(k,t) = \sum_{j=1}^{t} c_j I\{|R_{ij}-j| \leq k\}$, c_j's are constants and $I\{|R_{ij}-j| \leq k\}$ is 1 or 0 according as $|R_{ij}-j| \leq k$ or $|R_{ij}-j| > k$. The case $k = 0$, $c_j = 1$ corresponds to the classical matching problem (see Feller

[1], page 90). If the alternative is true, one expects that R_{ij} will be closer to j. Hence large M^b should support the alternative. The so-called "regression" constants $\{c_j, j=1,2...t\}$ are chosen to reflect the trend in the alternative. Though the above test is a within-block linear rank test, it is not a simple linear rank test in the sense of Hájek and Sidak (see [2], page 61).

In section 2, we study the asymptotic behavior of the above M-test, both under H_0 and under a contiguous sequence of alternatives. In Section 3 the sequential analog of the above problem, namely repeated significance testing, is developed and the average sample number is computed asymptotically under both sets of hypotheses. In section 4, we compare the M-test with the Page test, through asymptotic relative efficiency (ARE), for various values of t and with different underlying distribution functions F. Comparisons are also made through simulated powers and strength and weaknesses of the M-test are discussed.

2. THE TEST PROCEDURE AND ITS ASYMPTOTIC DISTRIBUTION

The following theorem gives the limiting distribution of the M-test under H_0.

<u>Theorem 2.1</u>. Under H_0, $\frac{M^b - b\mu_0}{\sqrt{b}\,\sigma_0}$ converges in distribution to the standard normal as $b \to \infty$ where

$$\mu_0 = E_0(M_1) = \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 1\le\alpha\le t}}^{j+k} t^{-1} c_j$$

and

$$\sigma_0^2 = \mathrm{Var}_0(M_1) = \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 1\le\alpha\le t}}^{j+k} t^{-1} c_j^2 + 2 \sum_{i<j}^{t} \sum^{t} \sum_{u=j-k}^{j+k} \sum_{\substack{v=i-k \\ 1\le u\ne v\le t}}^{i+k} \frac{c_i c_j}{t(t-1)} - \mu_0^2.$$

<u>Proof</u>:

Since M_i's are i.i.d. random variables with $\mathrm{Var}_0(M_1) < \infty$, by the central limit theorem, we have the asymptotic normality. Computation of

$E_0(M_1)$ and $Var_0(M_1)$ are as follows:

$$E_0(M_1) = \sum_{j=1}^{t} c_j E_0 I\{|R_{1j}-j|) \le k\}$$

$$= \sum_{j=1}^{t} c_j P_0 (j-k \le R_{1j} \le j+k)$$

$$= \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 1\le\alpha\le t}}^{j+k} t^{-1} c_j$$

where P_0 is the probability measure under H_0. The null variance is easily obtained as

$$\sigma_0^2 = E_0[\sum_{j=1}^{t} c_j I\{|R_{1j}-j| \le k\}]^2 - \mu_0^2$$

$$= E_0[\sum_{j=1}^{t} c_j^2 I[|R_{ij}-j| \le k\}]$$

$$+ 2E_0[\sum_{i<j}^{t} \sum^{t} c_i c_j I\{|R_{ij}-j| \le k\} I\{|R_{ij}-i| \le k\}] - \mu_0^2 \qquad (1)$$

$$= \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 1\le\alpha\le t}}^{j+k} t^{-1} c_j^2 + 2 \sum_{i<j}^{t} \sum^{t} \sum_{u=j-k}^{j+k} \sum_{\substack{v=i-k \\ 1\le u\ne v\le t}}^{i+k} c_i c_j P_0(R_{1j}=u, R_{1i}=v) - \mu_0^2 .$$

The proof is complete by noting $P_0(R_{1j}=u, R_{1i}=v) = \frac{1}{t(t-1)}$ for $i \ne j$ and $u \ne v$. ◫

Remark 1: As a consequence of Theorem 2.1, an appropriate test of level α is given by rejecting H_0 if $\frac{M^b - b\mu_0}{\sqrt{b}\,\sigma_0} \ge z_\alpha$, where z_α is the $100(1-\alpha)$ percentile of the standard normal distribution.

To study the asymptotic power properties of the M-test, consider the following sequence of contiguous alternatives

$$H_b: d_1/\sqrt{b} \le d_2/\sqrt{b} \le \dots \le d_t/\sqrt{b},$$

with at least one strict inequality. We also assume that F has a continuous density f on the support of F and $\int_{-\infty}^{\infty} f^2(x)dx < \infty$. The next theorem discusses the asymptotics of the M–test under H_b.

Theorem 2.2. Under the alternative H_b, $\dfrac{M^b - b\mu_o}{\sqrt{b}\,\sigma_o}$ converges in distribution to the normal distribution with mean θ and variance 1 as $b \to \infty$, where

$$\theta = \sigma_o^{-1} t \sum_{j=1}^{t} c_j(d_j - \bar{d}) \{\beta_{j-k-2,t-2} - \beta_{j+k-1,t-2}\},$$

with

$$\bar{d} = t^{-1} \sum_{j=1}^{t} d_i$$

and

$$\beta_{\alpha,t-2} = \begin{cases} \binom{t-2}{\alpha} \int_{-\infty}^{\infty} [F(x)]^{\alpha}[1-F(x)]^{t-2-\alpha} f^2(x)dx & \text{if } 0 \le \alpha \le t-2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof:

Let μ_b and σ_b^2 denote respectively the expected value and the variance of M_1 under H_b. It can be easily seen that σ_b^2 converges to σ_o^2 as $b \to \infty$. Under H_b, again using the central limit theorem, $\dfrac{M^b - b\mu_b}{\sqrt{b}\,\sigma_b}$ converges in distribution to the standard normal distribution as $b \to \infty$. Since $\dfrac{M^b - b\mu_o}{\sqrt{b}\,\sigma_o} = \dfrac{M^b - b\mu_b}{\sqrt{b}\,\sigma_b} \cdot \dfrac{\sigma_b}{\sigma_o} + \dfrac{\sqrt{b}}{\sigma_o}(\mu_b - \mu_o)$, to prove the theorem we only need to show that $\sqrt{b}(\mu_b - \mu_o)$ converges to $\theta\sigma_o$ as $b \to \infty$. If P_b denotes the measure under H_b, we have

$$\mu_b = E_b(M_1) = E_b[\sum_{j=1}^{t} c_j I\{|R_{1j}-j| \le k\}]$$

$$= \sum_{j=1}^{t} c_j P_b (j-k \le R_{1j} \le j+k). \tag{2}$$

Now $P_b(R_{1j} = \alpha+1) = P_b (\sum_{\substack{i=1 \\ i \neq j}}^{t} \eta_{ij} = \alpha)$,

where $\eta_{\ell j} = \begin{cases} 1 & \text{if } x_{1\ell} \le x_{1j}, \\ 0 & \text{otherwise} \end{cases}$.

Note that conditioned on $X_{1j} = x$, $\eta_{\ell j}$'s, $\ell = 1,2...j-1,j+1,...t$, are independent Bernoulli r.v.'s with parameter $p_{\ell j} = F(x-d_\ell/\sqrt{b})$. Since $\eta_{\ell j}$'s depend on the difference $(x_{1\ell}-x_{1j})$, the block effect b_1 gets cancelled out. Let $A = \{0,1\}$ and $A^{t-1} = AxA \dots (t-1)$ times. Let $i^\alpha = (i_1,i_2,\dots,i_{t-1}) \in A^{t-1}$ be such that $\sum_{j=1}^{t-1} i_j = \alpha$. Let I_α denote the set of all possible i^α's. Clearly I_α is a subset of A^{t-1} with cardinality $\binom{t-1}{\alpha}$. Thus

$$P_b(\sum_{\substack{\ell=1 \\ \ell \neq j}}^{t} \eta_{\ell j} = \alpha)$$

$$= \sum_{i^\alpha \in I_\alpha} \int_{-\infty}^{\infty} \prod_{\{\ell: i_\ell=1\}} F(x-d_\ell/\sqrt{b}) \prod_{\{\ell: i_\ell=0\}} [1-F(x-d_\ell/\sqrt{b})]\, dF(x-d_j/\sqrt{b})$$

$$= \sum_{i^\alpha \in I_\alpha} \int_{-\infty}^{\infty} \prod_{\{\ell: i_\ell=1\}} F[x-(d_\ell-d_j)/\sqrt{b}] \prod_{\{\ell: i_\ell=0\}} \{1-F[x-(d_\ell-d_j)/\sqrt{b}]\} dF(x). \tag{3}$$

Since F has a continuous density, we have the following Taylor expansion,

$$F[x-(d_\ell-d_j)/\sqrt{b}] = F(x)-(d_\ell-d_j)f(x)/\sqrt{b} + o_x(1/\sqrt{b}), \tag{4}$$

where $o_x(1/\sqrt{b})$ goes to zero as b goes to infinity when multiplied by $\sqrt{b}$.

Also note that

$$\mu_0 = \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 1\le\alpha\le t}}^{j+k} t^{-1} c_j = \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k \\ 0\le\alpha\le t-1}}^{j+k} c_j \binom{t-1}{\alpha} \int_{-\infty}^{\infty} F(x)[1-F(x)]^{t-1-\alpha} dF(x). \tag{5}$$

Combining (2), (3), (4) and (5) we have

$$\sqrt{b}(\mu_b - \mu_0)$$

$$= \sum_{j=1}^{t} \sum_{\substack{\alpha=j-k-1 \\ 0\le\alpha\le t-1}}^{j+k-1} c_j \sum_i \alpha \Big\{ \sum_{\{\ell: i_\ell=0\}} (d_\ell - d_j) \int_{-\infty}^{\infty} [F(x)]^{\alpha}[1-F(x)]^{t-2-\alpha} f(x)dF(x) \tag{6}$$

$$- \sum_{\{\ell: i_\ell=1\}} (d_\ell - d_j) \int_{-\infty}^{\infty} [F(x)]^{\alpha-1}[1-F(x)]^{t-1-\alpha} dF(x)\Big\} + o(1),$$

where $o(1)$ term goes to zero as $b \to \infty$. (Note that equation (4), Lebesgue dominated theorem and the assumption that $\int f^2(x)dx < \infty$ gives us the $o(1)$ term in the above expression.) Collecting terms in (6) we obtain the required result.

3. REPEATED SIGNIFICANCE TESTING

A repeated significance testing procedure is developed below as in Sen (see [12], page 243). Considerable savings in number of blocks b can be achieved if we perform the experiment sequentially. Let $U_b = M^b - b\mu_0$. Define $h_N = \text{Max}\{U_b : n_N \le b \le N\}$, where $n = n_N$ and N are the initial and target sample sizes respectively. For every $\alpha \in [0,1]$, n and N there exists an α_N and $h^o{}_N$ such that

$$0 \le \alpha_N = P(h_N > h_N^o | H_0) \le \alpha \le P(h_N \ge h_N^o | H_0).$$

Corresponding to the level of significance α and the critical value h_N^o, we may associate a stopping variable $\tilde{T} = \tilde{T}_N = \min\{b{:}n \leq N, U_b > h^o{}_N\}$ and $\tilde{T} = N$, if no such b exists. $U_{\tilde{T}}$ should be the statistic of interest, however, determining $h^o{}_N$ is almost impossible for large N and hence an appropriate α level test based on the invariance principle, is developed below.

Let $K_N(s) = i$ if $i N^{-1} \leq s < (i+1)N^{-1}$, $o \leq i < N$,

$= N$ if $s = 1$.

Note that $\{K_N(s),\ s \in [0,1]\}$ is a sequence of nondecreasing right continuous, nonnegative integer valued functions such that $K_N(1) = N$ and $K_N(0) = 0$. For every N, define a stochastic process $Y_N = \{Y_N(s), s \in [0,1]\}$ such that

$$Y_N(s) = N^{-1/2}\sigma_o^{-1}\, U_{K_N(s)},\quad 0 \leq s \leq 1.$$

Assume that $nN^{-1} \to 0$ as $N \to \infty$. Also, let $D_o^+ = \sup_{0 \leq s \leq 1} W(s)$, where $\{W(s){:}\ 0 \leq s \leq 1\}$ is the standard Brownian process. Let the upper $100\,\alpha$ percentile of D_o^+ be $D_o^+(\alpha)$. Since the M_i's are i.i.d. r.v.'s, under H_0, Y_N converges in distribution to W as $N \to \infty$ and hence

$$N^{-1/2}\sigma_o^{-1}h_N \to D_o^+ \tag{7}$$

An appropriate α level repeated significance test based on $\{U_b{:}n \leq b \leq N\}$ is as follows:

$$T = T_N = \min\{b{:}\ n \leq b \leq N,\ N^{-1/2}\sigma_o^{-1}U_b > D_o^+(\alpha)\},$$

$= N$, if no such b exists.

We reject H_0 if $N^{-1/2}\sigma_o^{-1}U_T > D_{t\,o}^+(\alpha)$ and accept it otherwise. The next theorem gives the average sample number ET asymptotically under H_0 as well as under H_b.

Theorem 3.1.

Under H_0,

$$\lim_{N\to\infty} N^{-1}ET = 1 - \sqrt{\frac{2}{\pi}} \int_0^1 \int_{D_0^+(\alpha)}^{\infty} t^{-1/2} \exp(-\frac{u^2}{2t}) du dt,$$

where as under H_b

$$\lim_{N\to\infty} N^{-1}ET = \int_0^1 P\{W(s) + \theta s \leq D_0^+(\alpha), \text{ for all } 0 \leq s \leq t\}\, dt,$$

and θ is as defined in Theorem 2.1.

Proof:

Note that under H_0,

$$N^{-1}ET_N = N^{-1}n + N^{-1}P(T=N) + N^{-1} \sum_{m=n}^{N-2} P(T > m)$$

$$= N^{-1}n + N^{-1}P(T=N) + N^{-1} \sum_{m=n}^{N-2} P(\max_{n \leq k \leq m} N^{-1/2}\sigma_0^{-1}U_k \leq D_0^+(\alpha)).$$

Since $N^{-1}n \to 0$ as $N \to \infty$, using (7) we get

$$\lim_{N\to\infty} N^{-1}ET = \int_0^1 P\{\sup_{0\leq s\leq t} W(s) \leq D_0^+(\alpha)\}\, dt. \tag{8}$$

It is well known that

$$P(\sup_{0\leq s\leq t} W(s) > x) = (\frac{2}{\pi t})^{1/2} \int_x^{\infty} \exp(-\frac{u^2}{2t})\, du. \tag{9}$$

Using the above identity and (8) gives us the first part of the theorem.

If $jN^{-1} \leq s < (j+1)N^{-1}$, under H_b we have,

$$E(Y_N(s)) = E\, N^{-1/2}\sigma_o^{-1}\, U_{k_N(s)}$$

$$= E\, N^{-1/2}\sigma_o^{-1}\, U_j$$

$$= E\, N^{-1/2}\sigma_o^{-1}(M^j - j\mu_b) + N^{-1/2}\sigma_o^{-1} j(\mu_b - \mu_o).$$

The above equation, (1) and $jN^{-1} \leq s < (j+1)\, N^{-1}$ imply

$$\lim_{N\to\infty} E\, Y_N(s) = s\theta.$$

Now convergence of the process Y_N under H_b, follows easily and the limiting process is $W + \mu^*$

where $\mu^*(s) = s\theta \qquad 0 \leq s \leq 1.$

Thus as in (8), average sample number under H_b is given by

$$\lim_{N\to\infty} N^{-1}ET = \int_0^1 P\{\sup_{0\leq s\leq t} [W(s) + \theta s] \leq D_o^+(\alpha)\}dt. \qquad (10)$$

Remark 2.

An expression parallel to (9) for $P\{\sup_{0\leq s\leq t} [W(s) + \theta s] \leq x\}$ is more complicated and only an infinite series is available. Also note that $1 - P\{\sup_{0\leq s\leq t} [W(s) + \theta s] \leq D_o^+(\alpha)\}$ is the asymptotic power of our sequential test under H_b.

4. DISCUSSION

Several test procedures are available in the literature for the proposed hypotheses. Each test has its merits and demerits and as mentioned in the introduction, they have been discussed by Pirie [7] and also by Puri and Sen [9]. It is well known that the Page's test attains its highest asymptotic power when the underlying distribution is Logistic whereas among–block rank tests of Hollander [3] and Puri and Sen [8] are more sensitive to the normal distribution. The Page's test is given by

$$W = \sum_{i=1}^{b} \sum_{j=1}^{t} (j - \frac{t+1}{2})(R_{ij} - \frac{t+1}{2}),$$

Under H_b, $\frac{W}{\sqrt{b}\,\sigma_W}$ converges in distribution to $N(\lambda,1)$ where

$$\sigma_W^2 = \frac{bt^2(t+1)^2(t-1)}{144},$$

$$\lambda^2 = t^2(t-1)\rho^2(d)[\int_{-\infty}^{\infty} f^2(x)dx]^2$$

and

$$\rho(d) = \frac{12}{t(t^2-1)} \sum_{j=1}^{t} (j - \frac{t+1}{2})d_j.$$

The ARE of the M–test with respect to the Page test is $e_{M,W} = \frac{\theta^2}{\lambda^2}$. The ARE's are calculated for various values of t and for variety of trends such as $d_j = \ln(j)$ and $d_j = j^3$. Computations indicate that the M–test is more sensitive to heavy–tailed distributions and the ARE increases with k. The values of $e_{M,W}$ are shown in table 4.1 for the Normal, Double Exponential, Logistic, Cauchy, and symmetric Pareto ($\alpha = .5$) distributions with $c_j = |j - \frac{t+1}{2}|$ and linear trend $d_j = j$. It may be recalled that Page's test is the locally most powerful rank test for this equi–spaced case i.e., the so–called linear trend, when the underlying distribution is the Logistic.

Appropriate choices of the c_j, $j = 1,2...t$, given the distribution F are obtained as follows: Observe that θ^2 in Theorem 2.2 can be written as $\frac{(U'C)^2}{C'AC}$, where $C = (c_1,c_2,...,c_t)'$, and prime denotes the transpose. The column vector U and positive definite matrix A can be found from the

Table 4.1

Asymptotic relative efficiencies of the M–test with respect to the Page test $(c_j = |j - \frac{t+1}{2}|,\ d_j = j)$

(t,k)	Standard Normal	Double Exponential	Logistic	Cauchy	Symmetric Pareto α=.5
(3.1)	.90	.90	.90	.90	.90
(4.1)	.76	.75	.79	.91	1.17
(5.1)	.73	.72	.76	.90	1.18
(5.2)	.76	.76	.78	.87	1.05
(6.1)	.68	.67	.70	.82	1.02
(6.2)	.72	.71	.76	.94	1.38
(7.1)	.63	.62	.65	.74	.85
(7.2)	.70	.69	.74	.95	1.49
(7.3)	.73	.72	.76	.89	1.19
(8.1)	.57	.56	.59	.65	.67
(8.2)	.68	.67	.72	.92	1.43
(8.3)	.71	.69	.75	.96	1.57
(9.1)	.53	.52	.54	.58	.53
(9.2)	.66	.64	.70	.88	1.28
(9.3)	.69	.68	.74	.98	1.74
(9.4)	.71	.70	.75	.91	1.32
(10.1)	.48	.48	.49	.51	.42
(10.2)	.63	.62	.67	.83	1.10
(10.3)	.68	.66	.73	.98	1.76
(10.4)	.70	.69	.75	.98	1.71
(11.1)	.44	.44	.45	.46	.34
(11.2)	.61	.59	.64	.78	.93
(11.3)	.67	.65	.71	.96	1.66
(11.4)	.68	.67	.74	1.01	1.95
(11.5)	.70	.69	.74	.92	1.42

expressions of θ^2 and σ_0^2 given in Theorems 2.1 and 2.2 respectively. Supremum over all possible vectors C is achieved when $C = A^{-1}U$ (see C.R. Rao, [11], page 60). When F is Cauchy, the optimal regression constants c_j are very close numerically to $|j - \frac{t+1}{2}|$.

The ARE's reflect the asymptotic performance of the tests. To get some idea of the performance of the M–test for moderate sample sizes, simulations were performed which indicate that this test performs well for heavy tailed distributions. Table 4.2 gives the results for $b = 30$ and when

Table 4.2

Power simulation of the M–test and the Page test

$(C_j = |j - \frac{t+1}{2}|,\ d_j = j)$

(t,k)	Standard Normal M	Standard Normal Page	Double Exponential M	Double Exponential Page	Logistic M	Logistic Page	Cauchy M	Cauchy Page	Stable (β=0, α=0.5) M	Stable (β=0, α=0.5) Page
(3.1)	.02	.04	.00	.00	.00	.00	.00	.00	.00	.00
(4.1)	.09	.13	.02	.01	.00	.00	.05	.06	.05	.06
(5.1)	.23	.32	.09	.09	.00	.00	.07	.05	.16	.20
(5.2)	.28	.32	.08	.09	.00	.00	.05	.05	.23	.20
(6.1)	.37	.55	.23	.35	.02	.00	.09	.08	.15	.16
(6.2)	.39	.55	.21	.35	.01	.00	.09	.08	.21	.16
(7.1)	.56	.80	.41	.67	.05	.14	.17	.19	.21	.24
(7.2)	.62	.80	.45	.67	.07	.14	.17	.19	.30	.24
(7.3)	.61	.80	.47	.67	.12	.14	.17	.19	.23	.24
(8.1)	.78	.97	.65	.92	.18	.31	.30	.34	.30	.30
(8.2)	.85	.97	.66	.92	.19	.31	.36	.34	.44	.30
(8.3)	.83	.97	.73	.92	.24	.31	.32	.34	.33	.30
(9.1)	.87	1.00	.66	.92	.36	.57	.46	.55	.37	.44
(9.2)	.96	1.00	.78	.92	.43	.57	.52	.55	.63	.44
(9.3)	.98	1.00	.87	.92	.47	.57	.57	.55	.63	.44
(9.4)	.96	1.00	.80	.92	.47	.57	.46	.55	.43	.44
(10.1)	.95	1.00	.81	1.00	.49	.81	.59	.78	.42	.65
(10.2)	.99	1.00	.94	1.00	.67	.81	.79	.78	.73	.65
(10.3)	1.00	1.00	.97	1.00	.69	.81	.83	.78	.83	.65
(10.4)	1.00	1.00	.98	1.00	.69	.81	.78	.78	.80	.65
(11.1)	.98	1.00	.98	1.00	.75	.97	.73	.93	.51	.82
(11.2)	1.00	1.00	.99	1.00	.89	.97	.91	.93	.82	.82
(11.3)	1.00	1.00	1.00	1.00	.92	.97	.97	.93	.93	.82
(11.4)	1.00	1.00	1.00	1.00	.93	.97	.95	.93	.94	.82
(11.5)	1.00	1.00	1.00	1.00	.91	.97	.91	.93	.78	.82

the underlying distributions are Cauchy, Normal, Double Exponential, Logistic, and symmetric Pareto $(\alpha = .5)$. (Simulation results are based on 300 replications for each value).

5. CONCLUDING REMARKS

The proposed M–test is particularly suited for heavy tailed distributions. It is a simple test to apply. Efficiency calculations and power simulations indicate that this test does much better than other existing tests, in particular the Page test. As seen from Tables 4.1 and 4.2, the three distributions Normal, Double Exponential, and Logistic behave almost the same since the tail properties are approximately the same. On the other hand, for Cauchy and symmetric Pareto ($\alpha = 0.5$) distributions the efficiencies and the simulated power increases and gets considerably better since these distributions have heavier tails than the first three. The M test appears to perform better with 10 or more treatments for moderately tailed distributions (like the normal) while it does better with a much smaller number of treatments for heavy tailed distributions (like the Pareto). Also, empirically, one may note that the best choice of k, the "window length" is $(\frac{t}{2} - 1)$ for t even and $(t-3)/2$ for t odd.

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